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Scattering and Information

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SCATTERING AND INFORMATION

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CONTENTS

The paper discusses some aspects of scattering of electromagnetic radiation in inhomogeneous media from the point of view of information theory.

A theory is developed, based on a concrete model of a typical scattering experiment, which takes into account the finiteness of the scattering volume and the coherence characteristics of the radiation; as a conclusion, the speckle noise which is superimposed over the signal is brought into evidence. More exactly, it is shown that the spurious oscillations are due to the non-vanishing character of the convolution square of the inhomogeneity distribution function at the point of truncation in the primary space.

Subsequently, the structural information of scattering experiments is examined and the finite logon-content determined. The sampling interval is established and the resolution achievable in scattering measurements is deduced; also the means by which it can be improved are examined.

An analysis of the quantitative information obtainable is pursued and the desirability of filtering is suggested. Finally, the way in which the partial coherence effectively acts as a low-pass filter is studied.

1. Introduction

The aim of this paper is to discuss some aspects of scattering of electromagnetic radiation by amorphous inhomogeneous media from the point of view of information theory. The problems treated here, have already been analysed in a few previous articles (Ross 1968, 1969*a*, *b*) first from an ideal classical electromagnetic standpoint and then by taking into account the coherence characteristics of the radiation field. The relevance of their reconsideration and interpretation from the point of view of information theory‡ is immediate: besides the fact that, by drawing such a parallel, scattering determinations acquire a new perspective, this kind of analogy, between different fields which have a logically equivalent structure, can clarify phenomena in one discipline when corresponding events are understood in the other.

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‡ Although criticism has been raised against this name, it is being kept here because of its widely spread use; in this paper information theory designates what Gabor (1956*a*) defines as communication theory, i.e. structural theory and statistical theory.

As the theory of communication has received considerable attention during the last twenty years, its heuristic value having been acknowledged, the beneficiary would be, of course, the theory of scattering. In fact, the theory of information, leading to a new investigation of the efficiency of different methods of observation, as well as their accuracy and reliability, has already found wide applications in various fields: telecommunications, computing, pure physics and discussion of the fundamental process of scientific observation (Brillouin 1967). Its importance for optical problems has been made clear by the contributions of Gabor, Fellgett, Linfoot, Blanc-Lapierre, to name but a few.

The approach adopted here is justified and becomes obvious on surpassing the concept of scattering determination as a simple Fourier correspondence between primary and Fraunhofer space, which is valid only for an ideal situation. If one takes into consideration the conditions of a real experiment, one is led to the conclusion that exact results are impossible and it becomes appropriate to treat a scattering experiment as a means of obtaining information about the structure of a medium, to treat the angular distribution of intensity as a received signal which gives information about the distribution of inhomogeneity, rather than being its power spectrum. The information on the structure of the sample transmitted to the observer is always limited, because it has to be obtained through a communication channel of finite capabilities—the instrument. The fundamental question, therefore, which we shall try to answer in this paper is: what do we really observe in a scattering experiment?

The informational approach to scattering determinations was prompted by a phenomenon remarked during some light scattering experiments. In an attempt to raise the low levels of scattered radiation by using a laser as light source, it was noticed that the angular distribution of intensity becomes very noisy and, in fact, a speckled pattern was easily seen with the naked eye or recorded on a photographic plate. It was not the only occasion, nor the first time that this phenomenon attracted attention. The 'speckle effect' observed whenever light having a high degree of coherence is scattered by a medium exhibiting random inhomogeneity, had already been known for many years and discussed in several papers. The interpretation conveyed by various authors (for example, Langmuir 1963 or Suzuki & Hioki 1966) is that the cause of this phenomenon is the high coherence of the radiation. *Post hoc, ergo propter hoc* is, however, not valid logically. In fact, all theories of scattering in random inhomogeneous media published so far assume complete coherence (although this is not usually stated explicitly) and none of them predicts the occurrence of speckles. It is therefore probably pertinent here to re-examine the approach to this subject and to extend the treatment to real cases. This topic, developed in the following section, is based on and represents an improved version of the theory first submitted in Ross (1969 *b*).

2. The real scattering problem

The problem of scattering of radiation by a random inhomogeneous medium studied here is the model of a concrete situation, corresponding to a typical scattering experiment.

Let us consider (see figure 1) an incoherent source of radiation (1), which is usually a secondary source,† i.e. the image of a primary source, formed by means of a condenser. We shall assume that the incoherent source is circular, with a diameter $\mathscr{D}_1$ and symmetric about the axis Ox. A collimator (2) with a focal distance $\mathscr{F}$, in whose focal plane the secondary source is situated,

† The conditions under which a secondary source acts effectively as an incoherent source are discussed by Born & Wolf (1959) and are normally fulfilled in a usual scattering experiment (Ross 1969 *b*).

produces a nearly parallel beam, having a (geometrical) divergence $\mathscr{D}_1/2\mathscr{F} \ll 1$. The sample (3) is contained between the planes $x = 0$, $x = \Delta l$. Its inhomogeneity will be described by the local variation in permittivity from the average

$$\delta\epsilon(\boldsymbol{r}) = \epsilon(\boldsymbol{r}) - \langle\epsilon\rangle,$$

where $\boldsymbol{r}$ is the positional vector defining the oriented distance from an arbitrary origin to a generic point, and

$$\langle\epsilon\rangle = \frac{1}{V}\iiint_V \epsilon(\boldsymbol{r})\,\mathrm{d}V_r$$

is the average permittivity over the volume V of the sample, or any part of it, as the medium is assumed to be statistically (i.e. macroscopically) homogeneous. The magnetic permeability will be supposed uniform and different from the value for vacuum by negligible amounts; the deductions will also be restricted to media with zero conductivity. For simplicity, the semi-spaces

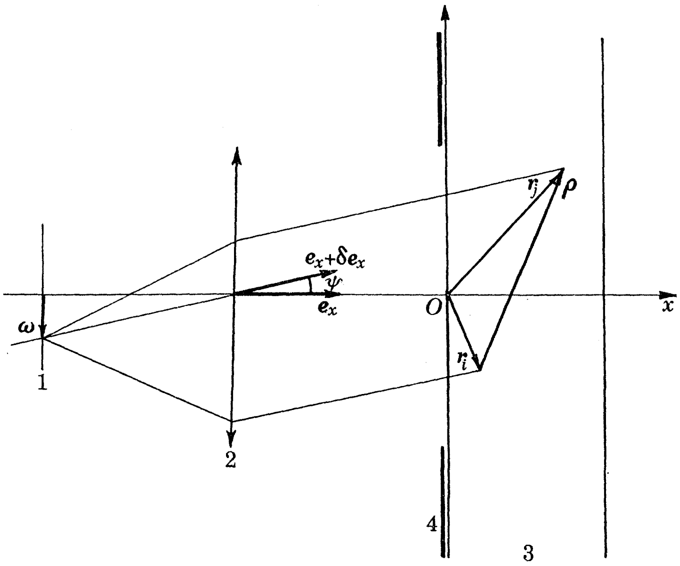

FIGURE 1

$x < 0$ and $x > \Delta l$ will be considered homogeneous, with a permittivity equal to $\langle\epsilon\rangle$. This avoids the additional complication of taking into account losses through reflexion at the planes $x = 0$, $x = \Delta l$, and corresponds to the normal experiment performed with the sample immersed in a liquid with matching refractive index. A diaphragm (4) in the plane $x = 0$ delimits the cross-section of the beam to a circle of diameter $\mathscr{D}$, symmetrical about the axis Ox. The criteria for choosing, $\mathscr{D}$, $\mathscr{D}_1$ and $\mathscr{F}$ are discussed in (Ross 1969*b*).

If the inhomogeneous medium is irradiated by a beam of light (or, in general, electromagnetic radiation), each element of volume $\mathrm{d}V_r$ becomes the source of a complex amplitude, described by its spectral density $\mathscr{E}(\boldsymbol{r}, t)$. The total scattered electric field $\boldsymbol{E}_s$ at an observation point described by $\boldsymbol{L} = \boldsymbol{m}L$ and at a certain time t, is the resultant of all elementary waves $\mathscr{E}$ produced at an anterior time $t(\boldsymbol{r}, \boldsymbol{L}; \nu)$, $t - t(\boldsymbol{r}, \boldsymbol{L}; \nu)$ being the time required by the radiation of frequency ν to travel from $\boldsymbol{r}$ to $\boldsymbol{L}$:

$$E_s(\boldsymbol{m}L, t) \propto \int_0^\infty \iiint_V \frac{\mathscr{E}(\boldsymbol{r}, t(\boldsymbol{r}, \boldsymbol{L}; \nu))}{|\boldsymbol{L} - \boldsymbol{r}|}\,\mathrm{d}V_r\,\mathrm{d}\nu. \tag{1}$$

For simplicity, the spectrum of the radiation will be assumed uniform with respect to the frequency and restricted to a narrow interval, this being the only case of real practical interest (quasimonochromaticity). The deductions will be restricted to weakly inhomogeneous media, i.e. those which do not distort the phase of the passing wave and alter its amplitude only slightly. In such circumstances (Ross 1968) the Born approximation is valid, and

$$\mathscr{E}(\boldsymbol{r}, t(\boldsymbol{r}, \boldsymbol{L}; \nu)) = \frac{E_0(\boldsymbol{r}, t(\boldsymbol{r}, \boldsymbol{L}; \nu))\, \delta\epsilon(\boldsymbol{r})}{2\Delta\nu},$$

where E_0 is the electric field of the incident wave and $\Delta\nu$ is the effective bandwidth of the spectral range of frequencies

$$\nu_0 - \Delta\nu \leqslant \nu \leqslant \nu_0 + \Delta\nu$$

small compared to the midfrequency

$$\frac{\Delta\nu}{\nu_0} \ll 1.$$

More exactly, taking into account the finite extent of the scattering volume, we have

$$\mathscr{E}(\boldsymbol{r}, t(\boldsymbol{r}, \boldsymbol{L}; \nu)) = \frac{E_0(\boldsymbol{r}, t(\boldsymbol{r}, \boldsymbol{L}; \nu))\, \delta\epsilon(\boldsymbol{r})}{2\Delta\nu}\, T(\boldsymbol{r}),$$

where $T(\boldsymbol{r})$ is a function which characterizes the geometry of the scattering volume. $T(\boldsymbol{r})$ has the value 1 inside the volume defined by the beam delimited by the diaphragm 4 in figure 1 and bounded by the planes $x = 0$, $x = \Delta l$ and vanishes identically outside it. (The amplitude of the electric field $|E_0(\boldsymbol{r}, t(\boldsymbol{r}, \boldsymbol{L}; \nu))|$ is zero outside the beam determined by the diaphragm and, as the semi-spaces $x < 0$ and $x > \Delta l$ were assumed homogeneous, at any point in them $\delta\epsilon(\boldsymbol{r}) \equiv 0$.) Such functions, which play a great role in structure analysis, have been introduced by Ewald (1940) and are called 'shape functions'.

As is well known, the field is not an observable magnitude; the measured quantity in a scattering determination is the intensity, defined by

$$\delta I(\boldsymbol{mL}) = \lim_{t\to\infty} \frac{1}{t} \int_0^t E_s(\boldsymbol{mL}, t)\, E_s^*(\boldsymbol{mL}, t)\, \mathrm{d}t, \tag{2}$$

the field being assumed stationary. The asterisk means complex conjugate. From here,

$$\delta I(\boldsymbol{mL}) \propto \frac{1}{2\Delta\nu} \int_{\nu_0-\Delta\nu}^{\nu_0+\Delta\nu} \iiint_V \frac{\mathrm{d}V}{|\boldsymbol{L}-\boldsymbol{r}_i|\,|\boldsymbol{L}-\boldsymbol{r}_j|} \iiint_V \lim_{t\to\infty} \frac{1}{t} \int_0^t E_0(\boldsymbol{r}_i, t(\boldsymbol{r}_i, \boldsymbol{L}; \nu))$$
$$\times E_0^*(\boldsymbol{r}_j, t(\boldsymbol{r}_j, \boldsymbol{L}; \nu))\, \delta\epsilon(\boldsymbol{r}_i)\, \delta\epsilon(\boldsymbol{r}_j)\, T(\boldsymbol{r}_i)\, T(\boldsymbol{r}_j)\, \mathrm{d}V\, \mathrm{d}t\, \mathrm{d}\nu, \tag{3}$$

The inner volume integral, giving the space average

$$\left\langle \lim_{t\to\infty} \frac{1}{t} \int_0^t E_0(\boldsymbol{r}_i, t(\boldsymbol{r}_i, \boldsymbol{L}; \nu))\, E_0^*(\boldsymbol{r}_j, t(\boldsymbol{r}_j, \boldsymbol{L}; \nu))\, \delta\epsilon(\boldsymbol{r}_i)\, \delta\epsilon(\boldsymbol{r}_j)\, T(\boldsymbol{r}_i)\, T(\boldsymbol{r}_j)\, \mathrm{d}t \right\rangle$$
$$= \left\langle \lim_{t\to\infty} \frac{1}{t} \int_0^t E_0(\boldsymbol{r}_i, t(\boldsymbol{r}_i, \boldsymbol{L}; \nu))\, E_0^*(\boldsymbol{r}_j, t(\boldsymbol{r}_j, \boldsymbol{L}; \nu))\, \delta\epsilon(\boldsymbol{r}_i)\, \delta\epsilon(\boldsymbol{r}_j)\, \mathrm{d}t \right\rangle \langle T(\boldsymbol{r}_i)\, T(\boldsymbol{r}_j) \rangle$$

can be written as

$$\left\langle \lim_{t\to\infty} \frac{1}{t} \int_0^t E_0(\boldsymbol{r}_i, t(\boldsymbol{r}_i, \boldsymbol{L}; \nu))\, E_0^*(\boldsymbol{r}_j, t(\boldsymbol{r}_j, \boldsymbol{L}; \nu))\, \mathrm{d}t \right\rangle \langle \delta\epsilon(\boldsymbol{r}_i)\, \delta\epsilon(\boldsymbol{r}_j) \rangle \langle T(\boldsymbol{r}_i)\, T(\boldsymbol{r}_j) \rangle, \tag{4}$$

if there is statistical independence between the electric field and the local inhomogeneity. Some authors (for example Bourret 1962*a*, *b*) consider the assumption of statistical independence between the perturbation and field function justified, and apply it whenever a two-point perturbation and the field occur as factors under a configuration integral, calling it local independence. Others classify such an approach as 'dishonest' (Keller 1962). Whether this hypothesis is valid in any circumstance is arguable; obviously, it can be safely made within the range of validity of the Born approximation (see also Beran & Parrent 1964, § 6.3; Molyneux 1968; Adomian 1963).

Within the brackets indicating the first average in the above expression, the cross-correlation of the fields at two space-time points is the well-known (Born & Wolf 1959, § 10.3) mutual coherence function $\Gamma(\mathbf{r}_i, \mathbf{r}_j, t(\mathbf{r}_j, \mathbf{L}; \nu) - t(\mathbf{r}_i, \mathbf{L}; \nu))$, equal, for quasimonochromatic fields, to (Beran & Parrent 1964, § 4.3):

$$\Gamma(\mathbf{r}_i, \mathbf{r}_j, t(\mathbf{r}_j, \mathbf{L}; \nu) - t(\mathbf{r}_i, \mathbf{L}; \nu)) = \exp(-2\pi i \nu_0 \tau_{ij})\, \Gamma(\mathbf{r}_i, \mathbf{r}_j, 0) = \exp(-2\pi i \nu_0 \tau_{ij})\, \mathscr{J}(\mathbf{r}_i, \mathbf{r}_j), \tag{5}$$

where

$$\tau_{ij} = t(\mathbf{r}_j, \mathbf{L}; \nu_0) - t(\mathbf{r}_i, \mathbf{L}; \nu_0)$$

and

$$\mathscr{J}(\mathbf{r}_i, \mathbf{r}_j) = \sqrt{I_0(\mathbf{r}_i)}\,\sqrt{I_0(\mathbf{r}_j)}\,\mu(\mathbf{r}_i, \mathbf{r}_j)$$

is the mutual intensity; $I_0(\mathbf{r}_i) = \mathscr{J}(\mathbf{r}_i, \mathbf{r}_i)$ is the intensity and μ the complex degree of coherence (coherence factor). As is well known, relation (5), in other words the quasimonochromatic approximation, is valid only for $|\tau|\,\Delta\nu \ll 1$. If the field is assumed to be statistically homogeneous (i.e. all the statistical moments are invariant with respect to a translation of the spatial variables), then

$$\sqrt{I_0(\mathbf{r}_i)} = \sqrt{I_0(\mathbf{r}_j)} = A_0$$

and

$$\mu(\mathbf{r}_i, \mathbf{r}_j) = \mu(\mathbf{r}_j - \mathbf{r}_i) = \mu(\boldsymbol{\rho}),$$

where

$$\boldsymbol{\rho} = \mathbf{r}_j - \mathbf{r}_i.$$

With this, relation (4) becomes

$$A_0^2 \exp[-2\pi i \nu_0 \tau]\, \mu(\boldsymbol{\rho})\, \langle T(\mathbf{r})\, T(\mathbf{r}+\boldsymbol{\rho})\rangle \langle \delta\epsilon(\mathbf{r}_i)\, \delta\epsilon(\mathbf{r}_j)\rangle. \tag{6}$$

It is easy to see that

$$2\pi\nu_0\tau = \langle k\rangle [\,|\mathbf{L}-\mathbf{r}_j| - |\mathbf{L}-\mathbf{r}_i|\,],$$

where

$$\langle k\rangle = \frac{2\pi}{\langle\lambda\rangle} = \frac{2\pi}{\lambda_0}\langle n\rangle = k_0\langle n\rangle$$

is the average wavenumber in the medium, whose average refractive index is $\langle n\rangle$. λ_0 is the wavelength in vacuum and $\langle\lambda\rangle$ is the average wavelength in the medium.

As

$$|\mathbf{L}-\mathbf{r}| = L - \mathbf{m}\mathbf{r} + \frac{1}{2L}[r^2 - (\mathbf{m}\mathbf{r})^2] - \dots,$$

if the size of the scattering volume is sufficiently small compared to the distance sample-observation point so that the phase shift introduced by the quadratic and higher order terms is considerably smaller than 1 rad, one obtains, retaining only the linear term in the above series of powers (Fraunhofer approximation†):

$$\exp[-2\pi i\nu_0\tau] = \exp[-\langle k\rangle i\mathbf{m}\boldsymbol{\rho}], \tag{7}$$

where

$$\boldsymbol{\rho} = \mathbf{r}_j - \mathbf{r}_i.$$

The second factor in (6), $\mu(\boldsymbol{\rho})$, the degree of coherence for the radiation field from an extended quasimonochromatic source, is given by the van Cittert–Zernike theorem (Born & Wolf 1959)

† The Fraunhofer approximation and its dependence on the degree of coherence has been examined in (Ross 1969 *b*).

as the average over the area of the source of the phase factor in the incident wave corresponding to two points, a distance $\boldsymbol{\rho} = \boldsymbol{r}_j - \boldsymbol{r}_i$ apart:

$$\mu(\boldsymbol{\rho}) = \frac{1}{\sigma_1}\int_{\sigma_1} \exp\left[\mathrm{i}\langle k\rangle (R_j - R_i)\right] \mathrm{d}\sigma_1,$$

where $\sigma_1 = \frac{1}{4}\pi\mathscr{D}_1^2$ is the area of the source, $\mathrm{d}\sigma_1$ an element of its surface at a generic point $\boldsymbol{\omega}$ and R_i and R_j are the distances from $\boldsymbol{\omega}$ to $\boldsymbol{r}_i$ and $\boldsymbol{r}_j$ respectively.

If $\boldsymbol{e}_x + \delta\boldsymbol{e}_x$ is the unit vector which characterizes the direction of propagation (making an angle ψ with the axis Ox of the system) of a wave originating at $\boldsymbol{\omega}$ it is easy to see that

$$R_j - R_i = \boldsymbol{\rho}(\boldsymbol{e}_x + \delta\boldsymbol{e}_x).$$

Let

$$\boldsymbol{r} = \boldsymbol{x} + \boldsymbol{n} = x\boldsymbol{e}_x + \boldsymbol{n},$$

where $\boldsymbol{x}$ is the component along the Ox axis and $\boldsymbol{n}$ the component in the plane yOz perpendicular to it, and similarly

$$\boldsymbol{\rho} = \boldsymbol{\xi} + \boldsymbol{\eta}.$$

In appendix I it is shown that, if the extent of the secondary source is sufficiently small compared to $\mathscr{F}$ so that

$$\tfrac{1}{2}\langle k\rangle (\boldsymbol{\rho}\boldsymbol{e}_x + \eta \tan\psi)\,\psi^2 \ll 1,$$

$$\mu = \langle\mu\rangle = \exp\left[\mathrm{i}\langle k\rangle \boldsymbol{\rho}\boldsymbol{e}_x\right] 2\,\frac{J_1(k_0\mathscr{D}_1\eta/2\mathscr{F})}{k_0\mathscr{D}_1\eta/2\mathscr{F}} = \exp\left[\mathrm{i}\langle k\rangle \boldsymbol{\rho}\boldsymbol{e}_x\right] |\mu|, \tag{8}$$

where J_1 is the symbol for Bessel function of order 1.

The complex degree of coherence μ is thus a product of two factors. The first, $\exp[\mathrm{i}\langle k\rangle \boldsymbol{\rho}\boldsymbol{e}_x]$, represents the phase factor and it is determined by the direction of the axis of the system; $\langle k\rangle \boldsymbol{\rho}\boldsymbol{e}_x$ is the phase difference between the waves from the centre of the source to the points $\boldsymbol{r}_i$ and $\boldsymbol{r}_j$ respectively. The second factor is the modulus of the complex degree of coherence; it alone characterizes the geometry of the collimation system and depends on the area of the source and focal distance of the collimator.

The third factor in (6) represents the average

$$\mathscr{T}(\boldsymbol{\rho}) = \frac{1}{V}\iiint_V T(\boldsymbol{r})\, T(\boldsymbol{r}+\boldsymbol{\rho})\,\mathrm{d}V.$$

$\mathscr{T}$ is a function of the kind known as convolution. In this case it is the convolution square of the shape function over the irradiated volume V. If, as a first approximation, one considers the volume V cylindrical, in other words one neglects the finite extent of the secondary source:

$$T(\boldsymbol{r}) = T(\boldsymbol{n})\, T(\boldsymbol{x})$$

and

$$\overset{2}{\widetilde{T(\boldsymbol{r})}} = \overset{2}{\widetilde{T(\boldsymbol{n})}}\, \overset{2}{\widetilde{T(\boldsymbol{x})}} = \iint_\sigma T(\boldsymbol{n})\, T(\boldsymbol{n}+\boldsymbol{\eta})\,\mathrm{d}\sigma \int_0^{\Delta l} T(\boldsymbol{x})\, T(\boldsymbol{x}+\boldsymbol{\xi})\,\mathrm{d}x$$

represents the convolution square of the shape function over the scattering volume (assumed cylindrical). In appendix II it is shown that

$$\mathscr{T}(\eta) = \frac{1}{\sigma}\overset{2}{\widetilde{T(n)}} = \frac{2}{\pi}\arccos\frac{\eta}{\mathscr{D}} - \frac{2}{\pi}\frac{\eta\sqrt{(\mathscr{D}^2-\eta^2)}}{\mathscr{D}}$$

and

$$\mathscr{T}(\xi) = \frac{1}{\Delta l}\overset{2}{\widetilde{T(x)}} = 1 - \frac{\xi}{\Delta l},$$

so that
$$\mathscr{T}(\boldsymbol{\rho}) = \frac{1}{V}\overset{2}{\widetilde{T(\boldsymbol{r})}} = \left(\frac{2}{\pi}\arccos\frac{\eta}{\mathscr{D}} - \frac{2}{\pi}\frac{\eta\sqrt{(\mathscr{D}^2-\eta^2)}}{\mathscr{D}^2}\right)\left(1-\frac{\xi}{\Delta l}\right). \tag{9}$$

The graph of the function $\mathscr{T}(\eta)$ is given in figure 2 and that of $\mathscr{T}(\xi)$ in figure 3. As the divergence of the beam has been neglected, for η approaching $\mathscr{D}$ an increasing relative error may be expected.

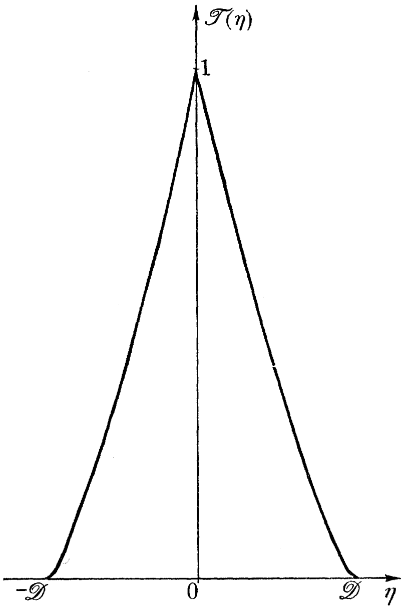

Figure 2

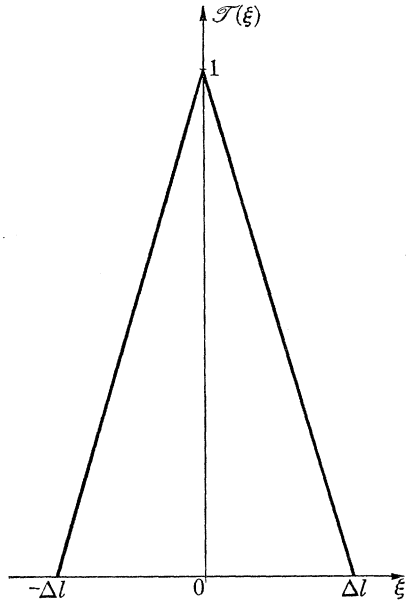

Figure 3

Finally, the last factor in (6) is, on the assumption that the field of the local deviation in permittivity from the average is statistically homogeneous,

$$F'_V(\boldsymbol{\rho}) = \langle \delta\epsilon(\boldsymbol{r})\,\delta\epsilon(\boldsymbol{r}+\boldsymbol{\rho})\rangle = \frac{1}{V}\iiint_V \delta\epsilon(\boldsymbol{r})\,\delta\epsilon(\boldsymbol{r}+\boldsymbol{\rho})\,\mathrm{d}V.$$

$F'_V(\rho)$ is also, after abstraction of the factor $1/V$, a convolution square. If the deviations in permittivity from the average are *random*, the convolution square over the scattering volume is a statistical average. For *infinite* volumes, it is identical to a well-known statistical second-order moment, namely the autocovariance, defined as

$$F'(\boldsymbol{\rho}) = \lim_{V\to\infty}\frac{1}{V}\iiint_V \delta\epsilon(\boldsymbol{r})\,\delta\epsilon(\boldsymbol{r}+\boldsymbol{\rho})\,\mathrm{d}V.$$

Some of the main properties of the autocovariance have been briefly outlined in Ross (1968) and are discussed in the specialized literature.† $F'(\rho)$ is maximum for $\rho = 0$

$$|F'(\boldsymbol{\rho})| \leqslant F'(0)$$

and
$$\lim_{\rho\to\infty} F'(\rho) = 0,$$

as can be easily visualized, if one translates the 'ghost' (see Hosemann & Bagchi 1962) with respect to the medium, because the deviations at $\boldsymbol{r}$ and $\boldsymbol{r}+\boldsymbol{\rho}$ quickly become independent and therefore the average of their product over an *infinite* volume is zero.

† An excellent exposition can be found in a recent book by Beran (1968), which presents in a unified manner the mathematical treatment relevant to various statistical continuum theories in different fields.

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This is, however, no longer true if the random medium has a finite extent. Displacing the 'ghost' a distance of the order of and approaching $\rho_{\max}$ (the maximum extent of the volume V), the average product of the deviations may not vanish in the ever diminishing volume available for the average. If the extent of the volume V is much larger than the distance over which the deviations in permittivity are correlated, i.e. if

$$a \ll \rho_{\max},$$

where a, the correlation distance, is a measure of the average distance over which the deviations in permittivity from the average are correlated, then

$$F'_V(\boldsymbol{\rho}) = F'(\boldsymbol{\rho}) + \langle\delta\epsilon^2\rangle f_V(\boldsymbol{\rho}), \tag{10}$$

where $$\langle\delta\epsilon^2\rangle = F'(0)$$

is the variance of the random field.

The correction term $f_V(\rho)$ describes the non-vanishing character, at large ρ, of the convolution square $F'_V(\boldsymbol{\rho})$ for finite volumes and it is a function of direction, because $\rho_{\max}$ is. $f_V(\rho)$ is negligibly small, practically zero, for all values of ρ except those very large, approaching $\rho_{\max}$. Therefore for $\rho \ll \rho_{\max}$,

$$F'_V(\boldsymbol{\rho}) = F'(\boldsymbol{\rho}).$$

Of course, the distortion due to the fact that, for ρ increasing, a smaller volume is available for the average, is accounted for by the weighting factor $\mathscr{T}(\rho)$.

Substituting the expressions for the factors in (6), the scattered intensity can be easily obtained. The factor of proportionality, accounting for the polarization characteristics of the electromagnetic field, was deduced in (Ross 1968) by solving a vector equation and is the same, in the range of validity of the quasimonochromatic approximation. Characterizing the scattering phenomenon by the cross-section (or Rayleigh ratio) R rather than intensity, the relation

$$R(s) = \frac{\langle k\rangle^4}{128\pi^2}\frac{\langle\delta\epsilon^2\rangle}{\langle\epsilon\rangle^2}(8-4s^2+s^4)\iiint_V F(\boldsymbol{\rho})\,\mathscr{T}(\boldsymbol{\rho})\,|\mu|\,(\eta)\exp[\mathrm{i}\langle k\rangle\,\boldsymbol{s\rho}]\,\mathrm{d}V$$
$$+\frac{\langle k\rangle^4}{128\pi^2}\frac{\langle\delta\epsilon^2\rangle}{\langle\epsilon\rangle^2}(8-4s^2+s^4)\iiint_V f_V(\boldsymbol{\rho})\,\mathscr{T}(\boldsymbol{\rho})\,|\mu|\,(\eta)\exp[\mathrm{i}\langle k\rangle\,\boldsymbol{s\rho}]\,\mathrm{d}V \tag{11}$$

is immediately obtained for unpolarized incident field. The Rayleigh ratio is defined as

$$R(s) = \frac{1}{\Delta l}\frac{\delta I(s)}{\phi_0},$$

where ϕ_0 is the flux in the incident wave,

$$\boldsymbol{s} = \boldsymbol{e}_x - \boldsymbol{m}$$

is the coordinate in the Fraunhofer space,

$$s = 2\sin\tfrac{1}{2}\alpha, \quad \alpha = \arccos(\boldsymbol{e}_x\,\boldsymbol{m})$$

and $$F(\boldsymbol{\rho}) = \frac{F'(\boldsymbol{\rho})}{\langle\delta\epsilon^2\rangle},$$

the normalized autocovariance, is known as the autocorrelation function of the random field.

Relation (11) describes the scattering phenomenon in inhomogeneous media of finite extent, taking into account the geometry of the collimating system, and it is correct within the range of validity of the quasimonochromatic approximation. The essential problem of the structure analysis is to determine the statistical parameters which describe the inhomogeneity, irrespective of the external shape of the scattering volume and of the coherence characteristics of the radiation field.

3. Information in scattering determinations

(a) *Noise in scattering patterns*

Let us assume first that the incident radiation is highly coherent over the cross-section of the beam, i.e. that practically $|\mu|(\eta) \equiv 1$ for $\eta \leqslant \mathscr{D}$. This can be achieved, for a given $\mathscr{D}$, either by diminishing the diameter of the secondary source, or by using the highly coherent radiation of a laser. The width of the autocorrelation function $F(\rho)$, given by the correlation distance a is, in general, much smaller than the width of either of the curves $\mathscr{T}(\eta)$ and $\mathscr{T}(\xi)$, which are $\frac{1}{2}\mathscr{D}$ (with an excellent approximation) and $\frac{1}{2}\Delta l$ respectively. If, neglecting the slight distortion introduced, one can assume that $\mathscr{T}(\boldsymbol{\rho}) \equiv 1$ for all ρ for which $F(\rho)$ is effectively non-zero (say $\rho \leqslant 2a$), then

$$F(\boldsymbol{\rho})\, \mathscr{T}(\boldsymbol{\rho}) \equiv F(\boldsymbol{\rho}). \tag{12}$$

In this case, for $F(\boldsymbol{\rho}) \equiv F(\rho)$ (i.e. isotropy of the random field), relation (11) becomes:

$$R(s) = \frac{\langle k\rangle^3}{32\pi}\frac{\langle \delta\epsilon^2\rangle}{\langle\epsilon\rangle^2}\frac{8-4s^2+s^4}{s}\int_0^\infty \rho F(\boldsymbol{\rho}) \sin(\langle k\rangle \rho s)\,\mathrm{d}\rho$$
$$+\frac{\langle k\rangle^4}{128\pi^2}\frac{\langle\delta\epsilon^2\rangle}{\langle\epsilon\rangle^2}(8-4s^2+s^4)\iiint_V f_V(\boldsymbol{\rho})\,\mathscr{T}(\boldsymbol{\rho}) \exp[\mathrm{i}\langle k\rangle \boldsymbol{s\rho}]\,\mathrm{d}V. \tag{13}†$$

Relation (13) states that, with the approximation (12), to the Fourier transform of the autocorrelation function, which is the angular spectrum of intensity for the ideal case of an infinitely large volume, a term is added, representing the contribution of the finiteness of the scattering volume. Neglecting the weighting effect of $\mathscr{T}(\rho)$, an approximate expression for the second term in (13) has been established‡ in (Ross 1969*b*):

$$C = \sum_{j=1}^{\infty} C_j = \sum_{j=1}^{\infty} \Omega_j \int_0^{\rho_{\max}} \rho^{j+1} \sin(\langle k\rangle \rho s)\,\mathrm{d}\rho$$
$$= \sum_{i=0}^{\infty}(-1)^i \Omega_{2i+1}\frac{(2j+2)!}{(\langle k\rangle s)^{2i+3}}\Big\{[\sin(\langle k\rangle \rho_{\max} s)]\sum_{j=0}^{i}(-1)^j\frac{(\langle k\rangle\rho_{\max} s)^{2j+1}}{(2i+1)!}$$
$$+[\cos(\langle k\rangle\rho_{\max} s)]\sum_{j=0}^{i+1}(-1)^j\frac{(\langle k\rangle\rho_{\max} s)^{2j}}{(2j)!}-1\Big\}$$
$$+\sum_{i=1}^{\infty}(-1)^i\Omega_{2i}\frac{(2i+1)!}{(\langle k\rangle s)^{2i+2}}\Big\{[\sin(\langle k\rangle\rho_{\max} s)]\sum_{j=0}^{i}(-1)^j\frac{(\langle k\rangle\rho_{\max} s)^{2j}}{(2j)!}$$
$$-[\cos(\langle k\rangle\rho_{\max} s)]\sum_{j=0}^{i}(-1)^j\frac{(\langle k\rangle\rho_{\max} s)^{2j+1}}{(2j+1)!}\Big\}. \tag{14}$$

† An inadvertence in (Ross 1969*b*) is hereby corrected (relation (14)).

‡ The formula (14) was established on the assumption that: $f_V(\rho)$ is isotropic, and that it can be expanded in a series of powers

$$f_V(\rho) = \sum_{j=1}^{i} \Omega_j \rho^j.$$

It can be easily seen that the first few terms in the series are ($\beta = \langle k \rangle \rho_{\max} s$):

$$C_1 = 2!\,\Omega_1 \rho_{\max}^3 \left[\frac{\cos\beta}{\beta^3} + \frac{\sin\beta}{\beta^2} - \frac{\cos\beta}{2!\,\beta} - \frac{1}{\beta^3}\right],$$

$$C_2 = -3!\,\Omega_2 \rho_{\max}^5 \left[\frac{\sin\beta}{\beta^4} - \frac{\cos\beta}{\beta^3} - \frac{\sin\beta}{2!\,\beta^2} + \frac{\cos\beta}{3!\,\beta}\right],$$

$$C_3 = -4!\,\Omega_3 \rho_{\max}^5 \left[\frac{\cos\beta}{\beta^5} + \frac{\sin\beta}{\beta^4} - \frac{\cos\beta}{2!\,\beta^3} - \frac{\sin\beta}{3!\,\beta^2} + \frac{\cos\beta}{4!\,\beta} - \frac{1}{\beta^5}\right],$$

$$C_4 = 5!\,\Omega_4 \rho_{\max}^6 \left[\frac{\sin\beta}{\beta^6} - \frac{\cos\beta}{\beta^5} - \frac{\sin\beta}{2!\,\beta^4} + \frac{\cos\beta}{3!\,\beta^3} + \frac{\sin\beta}{4!\,\beta^2} - \frac{\cos\beta}{5!\,\beta}\right],$$

Hence,

$$C = -\frac{\cos\beta}{\beta}\left[\Omega_1 \rho_{\max}^3 + \Omega_2 \rho_{\max}^4 + \Omega_3 \rho_{\max}^5 + \ldots\right]$$
$$+\frac{\sin\beta}{\beta^2}\left[\frac{2!}{1!}\Omega_1 \rho_{\max}^3 + \frac{3!}{2!}\Omega_2 \rho_{\max}^4 + \frac{4!}{3!}\Omega_3 \rho_{\max}^5 + \ldots\right]$$
$$+\frac{\cos\beta}{\beta^3}\left[\frac{2!}{0!}\Omega_1 \rho_{\max}^3\left(1 - \frac{1}{\cos\beta}\right) + \frac{3!}{1!}\Omega_2 \rho_{\max}^4 + \frac{4!}{2!}\Omega_3 \rho_{\max}^5 + \ldots\right]$$
$$-\frac{\sin\beta}{\beta^4}\left[\frac{3!}{0!}\Omega_2 \rho_{\max}^4 + \frac{4!}{1!}\Omega_3 \rho_{\max}^5 + \frac{5!}{2!}\Omega_4 \rho_{\max}^6 + \ldots\right] - \ldots$$

or

$$C = -A_1 \frac{\cos\beta}{\beta} + A_2 \frac{\sin\beta}{\beta^2} + A_3 \frac{\cos\beta}{\beta^3} - A_4 \frac{\sin\beta}{\beta^4} + \ldots.$$

As, in general, for all observable s,

$$\beta = \langle k \rangle \rho_{\max} s \gg 1,$$

it follows that

$$C \simeq -A_1 \frac{\cos\left(\langle k \rangle \rho_{\max} s\right)}{\langle k \rangle \rho_{\max} s}.$$

This represents a function oscillating rapidly around zero, with an interval between two successive maxima (or minima)

$$\Delta s = \langle \lambda \rangle / \rho_{\max}, \tag{15}$$

which describes the 'speckles' observed in experiments performed with highly coherent light. Relation (15) suggests that the aspect of the speckled pattern is a function of the shape of the scattering volume; this is in agreement with the qualitative observations reported by Rigden & Gordon (1962). In another paper discussing qualitatively the occurrence of the speckle effect, Oliver (1963) finds that the 'grain size' is inversely proportional to the extent of the scattering volume. This is indeed the essence of (15), and the interval expressed by it corresponds to the size of the speckles observed experimentally.

One may say, therefore, that the received signal, which is the scattered intensity measured by the Rayleigh ratio, is unavoidably affected by noise† (the second term in relation (13)), unless the scattering volume is infinitely large (or the wavelength infinitely small, but this is a trivial case corresponding to the limit of geometrical optics, at which the scattering phenomenon ceases to occur). The noise is superimposed over the signal, hence the received signal oscillates around the

† Following the terminology accepted in many fields of physics, Fellgett & Linfoot (1955) suggested the use of the term 'noise' in optics, to denote those fluctuations which, in the circumstances of a given experiment, must be regarded as unpredictable in detail and therefore a bar to perfectly exact measurements.

value corresponding to the ideal case of an infinitely large volume. For each s, the noise produces a region of uncertainty, which ultimately causes the error in determining $F(\rho)$.

The inevitability of errors, brought into pre-eminence by the epistemological conclusions of atomic physics, was adopted by the information theory as one of its basic principles, leading to a complete reappraisal of the importance of experimental errors. Accepting that every method of observation has its limitations, making experimental errors unavoidable, the problem is how to achieve the maximum efficiency in observation, in other words how to extract from an experiment the maximum possible amount of information about the parameters of interest, possibly by sacrificing information which is less relevant. But first, of course, one has to assess the amount of information obtainable from an experiment, i.e. the information content of a scattering spectrum. In fact, the generation of a smooth function $R(s)$ would imply an infinite amount of information per unit coordinate in the Fraunhofer space;† this requires, for specification, an infinite number of data. Therefore, if there is no limit to the subdivision of the coordinate in the Fraunhofer space, $R(s)$ could be described absolutely faithfully.

However, the region of incertitude produced by the noise around each point, equal in extent to the grain size of the speckles $\Delta s = \lambda/\rho_{max}$, leads to the observation that an interval $0-s_{max}$ cannot be meaningfully divided by more than $s_{max}/\Delta s = \rho_{max}\, s_{max}/\langle\lambda\rangle$ sampling points at which $R(s)$ may be measured (and as many points from which $F(\rho)$ will be determined). This conclusion, brought into evidence by Gabor (1946), constitutes the essence of the Whittaker–Shannon sampling theorem (Whittaker 1915; Shannon 1949) and can indeed be reached more rigorously, for example by expanding $R(s)$, defined in the interval $0-s_{max}$ in a Fourier series (or in terms of any set of orthogonal functions); the function $R(s)$ is assumed periodical, repeating indefinitely the behaviour of $R(s)$ between 0 and s_{max}. In the range of (spatial) coordinate in the primary space $0-\rho_{max}/\langle\lambda\rangle$, there are only $\rho_{max}\, s_{max}/\langle\lambda\rangle$ 'spectral' lines, all equally spaced by an interval $1/s_{max}$. Two data are associated with each line: the coefficients of the sine and cosine terms in the expansion; hence in the range of spatial coordinate $0-\rho_{max}/\langle\lambda\rangle$ there are $2\rho_{max}\, s_{max}/\langle\lambda\rangle$ independent Fourier coefficients.

Because, however, $R(s)$ is an even function ($F(\rho)$ is real), all the coefficients of the sine terms are identically zero and we are left therefore for a normal scattering experiment at frequencies corresponding to the visible region and higher, in which we measure only the intensity, with $\rho_{max}\, s_{max}/\langle\lambda\rangle$ independent Fourier coefficients. It must be mentioned, however, that for lower frequencies, at which the phase of the electric field is an observable magnitude, the sine coefficients are restored and the number of independent Fourier coefficients becomes $2\rho_{max}\, s_{max}/\langle\lambda\rangle$.

Here we shall restrict the treatment to the visible and X-ray regions of the electromagnetic spectrum, in which we are particularly interested and in which, in general, the phase cannot be detected. It ought to be added, perhaps, that even for visible radiation one can double the number of independent data available by taking into account the polarization characteristics of the scattered radiation; this can be indeed useful in studying anisotropic media (e.g. a semi-crystalline material), but here this problem will not be pursued.

As the amount of information obtainable from an experiment consists of the number of independent data (or can be defined as a function of it) which the instrument can extract from the signal, the above reasoning indicates that the number of independent dimensions or 'degrees

† With the assumption made, of highly coherent radiation over the cross-section of the incident beam. Later it will be shown that, with partially coherent radiation, one can obtain a smooth curve, although, of course, the amount of information per unit coordinate in the Fraunhofer space is not infinite.

of freedom' $\rho_{max} s_{max}/\langle\lambda\rangle$ (or, more exactly, the nearest smaller integer) can be regarded as a measure of the information supplied by the experiment. This conclusion was reached first by Gabor (1946) in his development of the communication theory.

However, the number of degrees of freedom does not characterize completely the information derived from an experiment; the number of degrees of freedom represents only one of the two features of the information content of a result, namely the structural information. The structural information consists of epistemological or *a priori* knowledge; it is *a priori* with respect to the experiment but, of course, not prior to the development of a plan of observation. Here, *a priori* means simply prior to the experiment in which we are interested; for our purpose, therefore, *a priori* is not necessarily taken in the strict Kantian sense, i.e. knowledge absolutely independent of all experience (Kant 1964, p. 43), but can be interpreted in a more relaxed sense (see, for example, Eddington 1939, p. 24; MacKay 1950), accepting that epistemological knowledge cannot be regarded as independent of observational experience altogether.

The structural information is provided by the knowledge of the experimental procedure and, as such, is, in a Kantian sense, transcendental;† more exactly, the structural information is determined (or rather limited) by the finite (differentiating) capacity of the instrument, in our case by the uncertainty introduced by the extent of the speckles. The examination of structural information, according to Kant (1964, p. 59), 'has not the purpose to extend knowledge, but only to correct it, and to supply a touchstone of the value, or lack of value, of all *a priori* knowledge'. The natural unit of structural information represents one independent datum, one elementary quantum of structural information; Gabor (1946) coined for it the name logon. The amount of structural information in a result, the logon content, is the number of independent categories or degrees of freedom which can be precisely defined.

The second feature of the information content is the quantitative or metrical information. It is *a posteriori* with respect to the experiment and it is provided by the conclusions derived from a study of the results of the observation, which have been obtained in a way described by the structural information. In fact, the experiment consists of associating a number, which defines the amount of evidence obtained, with each structural degree of freedom. This number describes the amount of metrical information obtained.

Let us discuss first the structural information of scattering determinations.

(*b*) *Structural information*

The result obtained, namely that the logon-content of a scattering experiment is finite and equal to

$$\mathcal{N} = \rho_{max} s_{max}/\langle\lambda\rangle,‡$$

or, more exactly,

$$\mathcal{N} \leqslant \rho_{max} s_{max}/\langle\lambda\rangle \tag{16}$$

since $\mathcal{N}$ is an integer, constitutes the essence of Gabor's (1956*a*, *b*, 1964) expansion theorem, which states basically that the structural information of a beam of light restricted both in its lateral and its angular extent is finite.§ Obviously, the expansion theorem is applicable, at least

† Kant (1964, p. 59) entitles transcendental 'all knowledge which is occupied not so much with objects as with the mode of our knowledge of objects in so far as this mode of knowledge is to be possible *a priori*'.

‡ The intervals $0-\rho_{max}$ and $0-s_{max}$ are sometimes called 'information intervals' in the primary space and Fraunhofer space respectively.

§ Identical to the number of independent eigensolutions of the wave equation in the finite domain defined (see also Miyamoto 1960).

approximately, to the phenomenon of scattering in inhomogeneous media. We must not forget that we are discussing a simplified version of the phenomenon; in fact, the speckled pattern is much more complicated, and appears as a superposition of effects due to all possible directions in the *three*-dimensional scattering volume.

Clearly, the logon-content of the experiment must be at least 1, hence

$$\rho_{\max} s_{\max}/\langle\lambda\rangle \geqslant 1 \qquad (17$$

puts into evidence the elementary area $\Delta\rho\Delta s/\langle\lambda\rangle = 1$ (= 1 logon) in an $s-\rho$ representation (Gabor's diagram of information) and reflects the quantal nature of the structural information communicable in a scattering experiment.

Relation (16) expresses the equivalence, from the point of view of structural information, of the interval explored in the Fraunhofer space and the maximum extent of the scattering volume, measured in wavelength units.

The structural information can be well described by the logon capacity of the experiment, defined by MacKay (1950) as the number of logons (degrees of freedom) per unit of coordinate interval. Thus, for a scattering determination, the logon capacity for the Fraunhofer space is:

$$\mathscr{L} = \rho_{\max}/\langle\lambda\rangle \quad \text{logons/unit coordinate in the Fraunhofer space,} \qquad (18)$$

or, more exactly, the nearest smaller integer.

The finite logon capacity of the experiment manifests itself in and determines the grain size of the speckles, i.e. the sampling interval in the Fraunhofer space. Comparing (15) with (18), we have

$$\mathscr{L} = 1/\Delta s.$$

One can also define a logon capacity for the primary space, which characterizes directly the resolution in determining $F(\rho)$:

$$\mathscr{L}' = s_{\max} \quad \text{logons/wavelength unit of coordinate in the primary space.} \qquad (19)$$

Obviously, $\mathscr{L}' \leqslant 2$, or, more exactly, $\mathscr{L}'$ can be, for a meaningful measurement, either 1 or 2 logons/wavelength. However, as will be seen below, this limit can be extended, at least for an inhomogeneous medium, whose inhomogeneity may best be treated from a statistical point of view.

Relation (18) expresses a principle, essential for scattering measurements, namely that the transmission of a certain amount of structural information per unit coordinate in the Fraunhofer space requires a certain minimum number of wavelengths in the extent of the scattering volume. Thus, for a given minimum Δs determined by the error in measuring the scattering angle, one cannot detect inhomogeneities on a scale larger than

$$\rho = \langle\lambda\rangle/\Delta s.$$

This suggests that, for making use of the whole (structural) information available, a minimum scattering volume (in wavelength units)

$$\frac{\rho_{\max}}{\langle\lambda\rangle} \geqslant \frac{1}{\Delta s}$$

is required. The condition is important for X-ray scattering experiments and determines, in fact, the 'resolving power' of the experimental arrangement.†

† For obvious reasons, in scattering experiments with X-rays, one is interested in extending *upwards* the scale of detail observable and therefore (in contrast to light-scattering experiments) the 'resolving power' customarily describes the *maximum* extent of the detail observable.

Conversely, for a given $\rho_{\max}$, the sampling interval Δs, i.e. the distance between two values of s at which $R(s)$ is measured, should not be chosen smaller than $\langle\lambda\rangle/\rho_{\max}$; to try to do so, in other words to attempt to talk of 'an interval Δs smaller than $\langle\lambda\rangle/\rho_{\max}$' would be to try to construct a logical pattern identical to that of 'an extent of the scattering volume larger than $\rho_{\max}$', which cannot, by definition, appear in any result and is therefore observationally meaningless (MacKay 1950). A Δs smaller than $\langle\lambda\rangle/\rho_{\max}$ cannot increase the amount of information retrievable from the measurement and a Δs larger than $\langle\lambda\rangle/\rho_{\max}$ entails a loss of information. $R(s)$ is completely determined by specifying its ordinates at a series of points spaced $\Delta s = \langle\lambda\rangle/\rho_{\max}$ apart. Therefore

$$\rho_{\max}\,\Delta s/\langle\lambda\rangle \geqslant 1. \tag{20}$$

Relation (20) defines a 'characteristic rectangle' in the diagram of information; the number of these rectangles is proportional to the logon content. (More exactly, the logon content is equal to the total area of the rectangles.)

The last expression leads to another observation.

As is well known (see, for example, Ross 1968), scattering measurements cannot provide local values of the inhomogeneous field investigated, yielding only average values; the average is taken over the scattering volume. If one is interested in local values of the inhomogeneous field $\rho_{\max}$ represents the uncertainty in locating the detail of interest, i.e. in ascribing its accurate position. Hence, it may be relevant (for example, if the medium presents a spatial variation of the mean characteristics of the inhomogeneity) to diminish the beam diameter and concentrate it on the detail of interest. In this case, of course, the grain size of the speckles unavoidably increases (for highly coherent radiation) or, what amounts to the same, the logon capacity decreases.†

Relation (20) suggests that there is a lower limit to the size of the scattering volume, i.e. a maximum accuracy with which one can establish the position of a detail; obviously, $\Delta s \leqslant 2$ and hence

$$\rho_{\max} \geqslant \tfrac{1}{2}\lambda.$$

One cannot possibly locate an inhomogeneity with an uncertainty smaller than half a wavelength.

It appears also that, in general, the minimum detail detectable is $\frac{1}{2}\langle\lambda\rangle$. Indeed, if one attempts to compute the Fourier transform of the recorded distribution of intensity, it is easy to see from the sampling or interpolation theorem (see, for example, Goldman 1953 or Arsac 1966) that the interval in the primary space (measured in wavelength units) between two points at which one should compute $F(\rho)$ is

$$\frac{\Delta\rho}{\langle\lambda\rangle} \leqslant \frac{1}{s_{\max}},$$

i.e. the maximum resolution achievable is

$$\Delta\rho = \frac{\langle\lambda\rangle}{s_{\max}} \leqslant \tfrac{1}{2}\langle\lambda\rangle,$$

in agreement, of course, with what was established above about the logon-capacity for the primary space (see relation (19)).

† Caution is required in using the so-called microbeam techniques (scattering determinations using beam diameters of the order of $50\,\mu$ and even lower. It must be emphasized that, for such experiments, relation (12) cannot automatically be applied.

Therefore, the thus computed (apart from a factor irrelevant to this discussion)

$$F\left(p\frac{\langle\lambda\rangle}{s_{\max}}\right) = \frac{1}{p}\frac{s_{\max}}{\rho_{\max}}\sum_{q=1}^{\mathscr{N}} q\frac{\langle\lambda\rangle}{\rho_{\max}} R\left(q\frac{\langle\lambda\rangle}{\rho_{\max}}\right)\sin\left(2\pi\frac{qp}{\mathscr{N}}\right),$$

with $p = 1, 2, \ldots, \mathscr{N}$, where (see relation (16))

$$\mathscr{N} = \frac{\rho_{\max}\, s_{\max}}{\langle\lambda\rangle}$$

(or, more exactly, the nearest smaller integer), differs from the ideal

$$F(\rho) = \frac{1}{\rho}\int_0^\infty sR(s)\sin\left(\langle k\rangle\rho s\right)\mathrm{d}s$$

at least in two important ways. First, the resolution obtainable, i.e. the smallest detail ascertainable, is inversely proportional to $s_{\max}$ (and hence, at best, to 2). Secondly, the calculated F shows, between sampling points, spurious oscillations, of period inversely proportional to $s_{\max}$, because of the truncation of the angular distribution of intensity, i.e. $R(s_{\max}) \neq 0$ (see Ross 1968, relations (60 to 61)). Besides, the values $R(q\Delta s)$ are affected by the (amplitude of the) speckles, due to the fact that the scattering volume is finite. Thus, the calculated $F(\rho)$ cannot coincide with the true $F(\rho)$ unless $R(s)$ is known from $s = 0$ to $s = \infty$ and the scattering volume is infinite.

Nevertheless, at least for media which can be described as randomly inhomogeneous (on a scale comparable to the natural unit in the primary space, which is the wavelength) the extent of the detail measurable can be much smaller than $\frac{1}{2}\langle\lambda\rangle$. Ideally, for randomly inhomogeneous media,† for which one can determine only the correlation functions which is a monotonic (decreasing) function, described, in general, by a very simple expression, there is no lower bound to the extent of the detail (correlation distance) ascertainable. In this case, one can overcome the finite resolution (i.e. $\mathscr{L}'$ limited by the relation $\mathscr{L}' \leqslant s_{\max}$) if, instead of performing the Fourier transform of the recorded intensity, a trial and error method is used, i.e. the recorded angular spectrum of intensity is compared with the Fourier transforms of the most usual types of correlation function, as indicated in (Ross 1969*a*). Indeed, in order to increase the resolution in determining $F(\rho)$ one needs, as it was shown above, to extend the interval of coordinate in the Fraunhofer space, available for determining $R(s)$, to values $s > 2$. The establishing, by trial and error, of the shape of $R(s)$, is equivalent to—and can be interpreted as—an extrapolation of $R(s)$ towards infinitely large values of s. Hence, ideally, there is no lower limit, due to the finite interval of values of s, to the extent of the detail, at least if it can be expressed by a correlation distance.

Naturally, by extrapolating the curve $R(s)$ to values $s > 2$, unforeseeable features, corresponding to finer scales of inhomogeneity, may be lost. Therefore, it is more correct to state that, ideally, there is no lower limit for the *dominant* scale of inhomogeneity but finer details (especially if the dominant correlation distance is smaller than $\frac{1}{2}\langle\lambda\rangle$) may be imperceptible. This is not necessarily a disadvantage; because many of the fine details (which are mostly unimportant in structure analysis) are lost, the observed functions $R(s)$ and $F(\rho)$ are much easier to treat quantitatively than the exact ones.

Nevertheless, in practice, there is a lower limit to the dominant detail discernible in this way (and, of course, to lower scales of detail).

The limit is imposed by the amplitude of the noise due to the finite size of the scattering volume.

† Also in the case of particulate scattering, if the shape of the scattering particle is known beforehand and with the condition of independent scattering.

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The amplitude of the noise can be larger than—and therefore can obscure—the difference between the ideal distribution of intensity (for an infinite scattering volume) corresponding to the scale of inhomogeneity investigated and the limit distribution for the extent of the inhomogeneity approaching zero corresponding to Rayleigh scattering.† The noise (from all sources) may also obscure the presence of other correlation functions, corresponding to lower scales of detail, if its amplitude is greater than the difference between the ideal distribution of intensity corresponding to the dominant correlation function and those corresponding to smaller scales of detail (and this is why they may not be detectable in the interval $s = (0, 2)$).

But, with this, we have started to discuss the amount of evidence posterior to the experiment, therefore we are in the domain of quantitative or metrical information.

(c) *Quantitative information*

The amplitude of the noise determines, in fact, the fineness of the scale on which the value of R at a sampling point (for each logon) is estimated, i.e. the amplitude of the noise establishes a graduation below which we cannot locate the result in one interval with a probability larger than 0.5. On such a scale, marked in 'minimum meaningful intervals', a magnitude is specified by the number of intervals which it occupies. MacKay (1950) coined the name metron for the unit of quantitative information: it is the quantitative information which enables one interval to be represented as occupied. The lower the amplitude of the noise, the smaller is an interval of the scale which it determines. The larger number of intervals thus occupied, corresponds to, in MacKay's terminology, a higher metron content. From here, it appears obvious that the metrical information has also a quantal nature.

The 'extent' of the noise, i.e. the graininess of the speckles, determines therefore the structural information, while the amplitude of the noise affects the quantitative information. However, if the extent of the speckles can be easily assessed, the amplitude cannot be, in general, calculated. The metrical information depends also on the amplitude of the signal, ultimately on the shape of the correlation function. But this can be established only posterior to the experiment.

Thus, in general, it is impossible to know, before an experiment is performed, how disturbing the amplitude of the noise will be, i.e. whether the metron content will allow a meaningful assessment of the relevant magnitudes. A few observations are nevertheless possible. For the same extent of the scattering volume, the coarser the scale of the inhomogeneity, the more disturbing the noise. Also, of course, the smaller the extent of the scattering volume, the lower the metron content. This is, in fact, the main obstacle in using microbeam techniques. The patchiness of scattering patterns recorded with microbeam techniques is illustrative (see, for example, Birnboim, Magill & Berry 1967) and casts some doubt on the reliability of such methods.

One can establish *a posteriori* the amount of quantitative information retrievable in a scattering experiment (or, at least, the bounds between which it can vary); a possibility of assessing it is, for example, Shannon's (1949) well-known 'capacity of the channel'.

Applying Goldman's (1953) unambiguous terminology to our problem, if the term liniva designates the value of the dependent variable (for example R) corresponding to a certain s, quadiva will denote the square of the liniva and the integral of the quadiva in the Fraunhofer space will be called the quadratic content.

† Of course, making abstraction of the finite sensitivity of the measuring instrument and also neglecting other sources of noise. These, however, start to play an important role only for ρ_{max} larger than the usual values in experiments. The limitation due to the finite extent of the scattering volume, in general, is predominant.

With this, if P is the average quadiva of the signal (i.e. of the Rayleigh ratio corresponding to an infinitely large volume) and N is the average quadiva of the noise, the capacity of the channel-scattering experiment is given by the formula

$$C \propto \rho_{\max} \lg \frac{P+N}{N}, \tag{21}$$

with a factor of proportionality depending on the units chosen. As is well known, the capacity of the channel can be achieved only by the use of infinitely long sequences, or, in our case, only if an infinite interval of values for s were available for recording the intensity.

The relation is valid only if the signal and noise in the system are independent (or incoherent), i.e. their average quadiva (or quadratic content) are additive; this, although probably not strictly true, can be safely accepted for our problem. However, (21) has been established and is valid only for white noise and it would be hazardous to assume this for speckleness. For arbitrary noise, the problem of determining the channel capacity seems insoluble explicitly, Shannon's (1949) theory providing only the upper and lower bounds for the maximum rate (per unit coordinate in the Fraunhofer space) of transmission of (quantitative) information in a noisy system.

It would be more interesting to establish generally valid criteria which could estimate the acceptable amplitude of noise in a particular case (i.e. for a given correlation function) but this, if it is possible, is very difficult. Without exception, noise reduction—or the separation of the signal from noise—is a fundamental desideratum.

The process of separation of signal and noise is called, in information theory, smoothing; if the separation is performed automatically by a piece of equipment, the smoothing process is called filtering and the equipment which performs the separation is called a filter.

One obvious way of reducing the specific weight of the amplitude of the noise is by increasing the scattering volume.† Nevertheless, the extent of the scattering volume cannot be increased indefinitely, if not for another reason, at least because the Fraunhofer requirement imposes soon impracticable distances between sample and observation point. Besides, the finite interval of electromagnetic frequencies sets, by the condition of quasimonochromaticity, an upper limit to the extent of the scattering volume, and also, of course, the smaller the scattering volume, the more exact the Born approximation (Frisch 1966, § 2, 1967, § 16).

There is, however, another possibility of extracting the signal from noise: by filtering. The noise which affects the measured angular spectrum of intensity is produced by the spurious $f_V(\rho)$ which, as established above, has practically non-zero values only for ρ of the order of $\rho_{\max}$. Of course, an accurate filter must remove the noise without distorting the signal; the perfect solution for cutting off the large values of ρ without affecting the small ones is an ideal low-pass filter with a rectangular spectrum

$$U(\rho) = \begin{cases} 1 & (\rho \leqslant \rho_0), \\ 0 & (\rho > \rho_0); \end{cases}$$

ρ_0 must obviously be $$a \ll \rho_0 < \rho_{\max}$$

† It is well known in communication theory that one can exchange bandwidth for signal/noise ratio; the noise improvement factor increases at least linearly with the bandwidth. Indeed, this effect, i.e. the reduction in the specific weight of the noise amplitude with the extension of the bandwidth, has been reported, for the *image plane* (i.e. primary space) in holography (or other image forming systems) by Enloe (1967) and Gerritsen, Hannan & Ramberg (1968). Their formulae and quantitative results, obtained assuming a model not valid to the problem under discussion, cannot be readily applied here. Nevertheless, the deductions in this paper are in agreement with, and follow the trend expressed in their conclusions.

where the correlation distance a estimates the average extent of the scale of inhomogeneity. With such a filter, the second term in (13) vanishes, while the first is left unchanged; the smoothed angular spectrum of intensity thus obtained corresponds to the ideal case of an infinite scattering volume. However, it is known in communication theory that, according to the Paley–Wiener criterion, such a filter with a sharp cut-off cannot exist.

There is, nevertheless, available a physically realizable low-pass filter: the partial coherence. Let us rewrite (11), substituting for $|\mu|\,(\eta)$ the expression given by (8):

$$R(s) = \frac{\langle k\rangle^4}{128\pi^2}\frac{\langle\delta\epsilon^2\rangle}{\langle\epsilon\rangle^2}(8-4s^2+s^4)\iiint_V F(\boldsymbol{\rho})\,\mathscr{T}(\boldsymbol{\rho})\,2\frac{J_1(k_0\mathscr{D}_1\eta/2\mathscr{F})}{k_0\mathscr{D}_1\eta/2\mathscr{F}}\exp\left[\mathrm{i}\langle k\rangle\,\boldsymbol{s\rho}\right]\mathrm{d}V$$
$$+\frac{\langle k\rangle^4}{128\pi^2}\frac{\langle\delta\epsilon^2\rangle}{\langle\epsilon\rangle^2}(8-4s^2+s^4)\iiint_V f_V(\boldsymbol{\rho})\,\mathscr{T}(\boldsymbol{\rho})\,2\frac{J_1(k_0\mathscr{D}_1\eta/2\mathscr{F})}{k_0\mathscr{D}_1\eta/2\mathscr{F}}\exp\left[\mathrm{i}\langle k\rangle\,\boldsymbol{s\rho}\right]\mathrm{d}V. \quad (22)$$

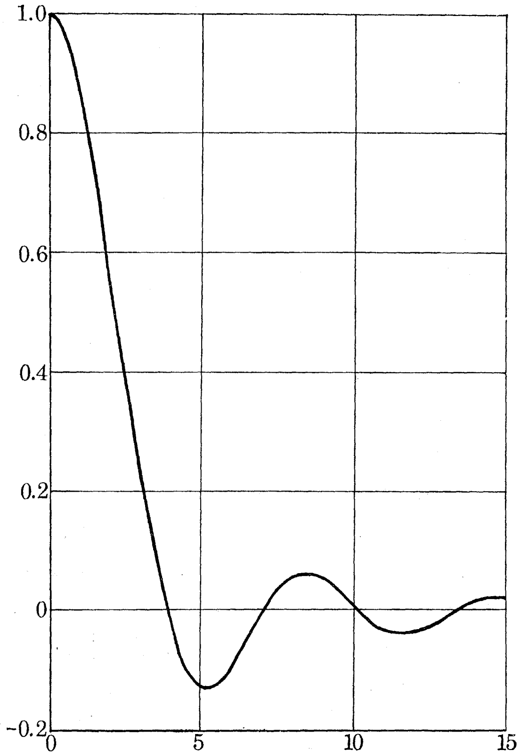

FIGURE 4

The graph of $|\mu|\,(\eta) = 2\dfrac{J_1(k_0\mathscr{D}_1\eta/2\mathscr{F})}{k_0\mathscr{D}_1\eta/2\mathscr{F}}$ against $k_0\mathscr{D}_1\eta/2\mathscr{F}$ is given in figure 4.

The width of this curve, i.e. the value of the argument $k_0\mathscr{D}_1\eta/2\mathscr{F}$ for which $\mu(\eta) = 0.5$ corresponds to

$$k_0\mathscr{D}_1\eta/2\mathscr{F} = 2$$

(more exactly 2.22) which means that the width of the curve corresponds to the coherence distance

$$\eta_{\mathrm{coh}} = 4\mathscr{F}/k_0\mathscr{D}_1. \quad (23)$$

The coherence distance η_{coh} is the effective† width of the low-pass filter.

† Interference effects do occur for values of coherence factor smaller than 0.5. The width of the filter represents only an average value.

An accurate filter does not distort the signal; for this, $|\mu|$ must be practically unity for all values of ρ for which $F(\rho) \neq 0$. The obvious condition which has to be fulfilled is

$$\eta_{coh} \gg a. \tag{24}$$

In order to have an efficient reduction of the noise, for a given ρ_{max}, $\mathscr{D}_1$ and $\mathscr{F}$ must be chosen so that

$$\eta_{coh} \ll \rho_{max}. \tag{25}$$

In practice, of course, a compromise must be reached. Using visible light, for example, a is in general smaller than a wavelength and a reasonable value for $\mathscr{D}$ is $1\,\mathrm{mm} \simeq 2000\lambda_0$; $\eta_{coh} \simeq 250\lambda_0$ is therefore a good compromise, reached for $\mathscr{D}_1/\mathscr{F} \simeq 25 \times 10^{-4}$.

Even if (24) is not fulfilled, as long as $a < \eta_{coh}$, the filter will transmit the signal, of course distorting it. It must be emphasized, however, that the filter will not transmit any information about inhomogeneity larger than the effective width. Like Eddington's (1939, p. 16) ichtyologist, we must not be surprised, therefore, if we do not find any inhomogeneity on a scale larger than η_{coh} (see also Gabor 1964). This is very important in practical determinations, when one does not know *a priori* the scale of inhomogeneity investigated, or if the sample presents several scales of inhomogeneity.

It must be realized that η_{coh} is, at any rate within the limit of validity of the quasimonochromatic approximation, a two-dimensional filter acting on a three-dimensional problem. The result is, nevertheless, extremely good for a judicious compromise, consisting of an efficiently smoothed angular spectrum of intensity.†

Assuming that

$$|\mu|\,(\eta)\,\mathscr{T}(\boldsymbol{\rho})\,F(\rho) \simeq F(\rho),$$

the relation

$$R(s) = \frac{\langle k\rangle^3}{32\pi}\frac{\langle\delta\epsilon^2\rangle}{\langle\epsilon\rangle^2}\frac{8-4s^2+s^4}{s}\int_0^\infty \rho F(\rho)\sin(\langle k\rangle\rho s)\,\mathrm{d}\rho$$
$$+\frac{\langle k\rangle^4}{128\pi^2}\frac{\langle\delta\epsilon^2\rangle}{\langle\epsilon\rangle^2}(8-4s^2+s^4)\iiint_V f_V(\rho)\,2\frac{J_1(k_0\mathscr{D}_1\eta/2\mathscr{F})}{k_0\mathscr{D}_1\eta/2\mathscr{F}}\,\mathscr{T}(\boldsymbol{\rho})\exp[\mathrm{i}\langle k\rangle\,\boldsymbol{s\rho}]\,\mathrm{d}V \tag{26}$$

is obtained, with a drastically diminished second term; no speckle effect can, in general, be observed in partially coherent radiation, which practically provides the ideal distribution of intensity corresponding to an infinite scattering volume.

The metron content of individual logons is therefore increased but, obviously, the logon content decreases;‡ the logon capacity of the experiment becomes

$$\mathscr{L} = \eta_{coh}/\langle\lambda\rangle.$$

It is not easy to establish the change in the total metron-content of the result and even more difficult to establish criteria. If we imagine, as suggested by MacKay (1950), the information-vector of length $\sqrt{i}$ (i = total metron content of the result) defined in a space having a number of dimensions equal to the logon content, the metron content of each logon being the square of the

† The reduction of the amplitude of the noise by decreasing the coherence of the radiation has been observed for holography and discussed by various authors (for example Bertolotti, Gori & Guattari 1967); from the point of view of image formation this problem is studied by Considine (1966) and some aspects are also outlined by Thomas (1968) (see also Thompson 1969). The coherence requirements in wavefront reconstruction have been investigated by various authors, starting with Gabor (1949, 1951). (See also Reynolds & De Velis 1967 and De Velis & Reynolds 1967).

‡ The loss in resolution with the reduction in coherence has been observed in holography (Reynolds & De Velis 1967).

projection of the information vector on the respective axis, the adjustment of ρ_{max} and η_{coh} corresponds to a rotation of the information vector and probably a change in its length.

The ultimate effect of the low-pass partial coherence filter consists of increasing the concentration of metrons in the logons which one really wants to observe, provided by the effective width of the filter.

By adjusting the *a priori* data, therefore, one can create the optimum compromise for the scattering measurement (although, as said above, it is difficult to establish criteria for this), in other words, one can improve the accuracy in measuring parameters of interest by ignoring others.

It may be mentioned here that the low-pass filter aspect of partial coherence can be inferred, for example, from the elegant experiment performed by Thompson & Wolf (1957). If their result is interpreted from the reciprocal point of view adopted here, one is easily led to the observation that, with a judicious choice of η_{coh} one can obtain information either on the individual pinholes or on their reciprocal position.

The effect of the size of the secondary source (i.e. of the coherence distance) upon the aspect of the speckled pattern was also beautifully illustrated by Hosemann & Bagchi (1962, chap. VI).

4. Conclusions

The considerations above enable one to answer the fundamental question posed at the beginning: what do we really observe in a scattering experiment?

It is clear from our deductions that, once the design of the (scattering) experiment is established, certain characteristics which any observation will present—and which will be discovered *a posteriori*—can be foreseen *a priori*, simply because the pre-established plan of observation will be employed.† The occurrence of speckles in highly coherent radiation, for example, is, from this point of view, *a priori* epistemological knowledge.

As in all similar problems, the spurious oscillations which, for scattering phenomena appear as speckles, are due to the non-zero values, of the function defined in the primary space, at the point of truncation, i.e. to the non-zero values of the convolution square of the inhomogeneity for $\rho = \rho_{max}$. However, the finite sampling interval in the Fraunhofer space, although equal to the grain size of speckles, is not due to the non-zero value of the function at the point of truncation, but only to the fact that a finite interval of variable in the primary space is available. Even if, by filtering, the spurious oscillations are suppressed, or at any rate, the speckle noise is reduced to such an extent that it practically disappears, the sampling interval remains finite; in fact, as has been shown above, its length increases.

Such effects are well known in, for example, Fourier transform spectroscopy, for which the commonly applied apodization technique is the mathematical equivalent of the use of partially coherent radiation in scattering determinations. Indeed, the shape of the low-pass partial coherence filter is similar to the apodizing function sinc (x/X) proposed by Strong & Vanasse (1959). One must not forget, however, that, because of the dimensional multiplicity of the primary space, the problem is more complicated for scattering experiments than for spectroscopic determinations.

The treatment developed above was based on the assumption, which corresponds to the typical scattering experiment, that $\rho_{max} \gg a$ and, in this case, it has been possible to express the average $F_V'(\rho)$ as a sum

$$F_V'(\boldsymbol{\rho}) = F'(\boldsymbol{\rho}) + \langle \delta\epsilon^2 \rangle f_V(\boldsymbol{\rho}).$$

† 'We can know *a priori* of things only what we ourselves put into them' (Kant 1964, p. 23).

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Obviously, if the extent of the scattering volume is of the same order of magnitude as the correlation distance, the above decomposition is no longer valid and $F(\rho)$ is no longer retrievable from the computed $F'_V(\rho)$ (or, more exactly, $F'_V(\rho)\,\mathscr{T}(\rho)$). It may be mentioned that, in this case, as can be easily seen from the convolution theorem, the minima of the speckles become zero. This explains the high amplitude of the speckles occurring with microbeam techniques and stresses the need for caution in such measurements. Taking into account that also the assumption (12) is, in general, not applicable, one is led to the formulation of what is probably a fundamental principle of scattering determinations: the extent and position of a detail can only be known with a mutually related uncertainty or, otherwise expressed, a combination of exact extent of a detail with its exact position is not observable.

The treatment carried out in this paper was intended as a contribution towards establishing a model which approximates better the conditions of a real experiment. Such ideal assumptions as the infinitely large character of the scattering volume or the infinitely small extent of the secondary source and spectral interval of electromagnetic frequencies have been eliminated from the theory of scattering determinations presented here.

The result obtained by taking into account the finite extent of the scattering volume and the coherence characteristics of the radiation field pointed to the observation that no scattering experiment is able to give results of infinite accuracy, no scattering experiment is able to provide an infinite amount of information. This conclusion, namely the finiteness of information available to man, has been singled out (van Soest 1956) as the most important principle of information theory.

Based on this principle, it has been possible to suggest the way in which one can increase the efficiency of scattering determinations, i.e. the amount of *relevant* information retrievable from an experiment. To achieve this, was the main object of the work presented here.

Appendix I

Calculation of the degree of coherence

$$\mathscr{J}(\boldsymbol{r}_i, \boldsymbol{r}_j) = \int_{\sigma_1} I(\boldsymbol{\omega})\, \frac{\exp[\mathrm{i}\langle k\rangle(R_j - R_i)]}{R_i R_j}\, \mathrm{d}\sigma_1,$$

$$\mu(\boldsymbol{r}_i, \boldsymbol{r}_j) = \frac{\mathscr{J}(\boldsymbol{r}_i, \boldsymbol{r}_j)}{\sqrt{I(\boldsymbol{r}_i)}\sqrt{I(\boldsymbol{r}_j)}}$$

where

$$I(\boldsymbol{r}) = \int_{\sigma_1} \frac{I(\boldsymbol{\omega})}{R^2}\, \mathrm{d}\sigma_1.$$

Hence,

$$\mu = \frac{1}{\sigma_1}\int_{\sigma_1} \exp[\mathrm{i}\langle k\rangle(R_j - R_i)]\, \mathrm{d}\sigma_1,$$

$$R_j - R_i = \boldsymbol{\rho}(\boldsymbol{e}_x + \delta \boldsymbol{e}_x) = \overline{AD} \quad \text{(see figure 5)}$$

where

$$\rho = \overline{AB}, \quad \eta = \overline{BC}, \quad \delta e_x = 2\sin\tfrac{1}{2}\psi, \quad x = \boldsymbol{\rho}\boldsymbol{e}_x = \overline{AC} = \overline{AG},$$

$$\boldsymbol{\rho}(\boldsymbol{e}_x + \delta \boldsymbol{e}_x) = \overline{AG} + \overline{GD} = \overline{AG} + \overline{HD} - \overline{HG}.$$

As

$$\overline{CG} = 2\boldsymbol{\rho}\boldsymbol{e}_x \sin\tfrac{1}{2}\psi,$$

$$\overline{HG} = 2\boldsymbol{\rho}\boldsymbol{e}_x \sin^2\tfrac{1}{2}\psi,$$

$$\overline{HD} = \overline{CM} = \eta\sin\psi,$$

$$\boldsymbol{\rho}(\boldsymbol{e}_x + \delta\boldsymbol{e}_x) = \boldsymbol{\rho}\boldsymbol{e}_x + \eta\sin\psi - 2\boldsymbol{\rho}\boldsymbol{e}_x\sin^2\tfrac{1}{2}\psi$$
$$= \boldsymbol{\rho}\boldsymbol{e}_x\cos\psi + \eta\sin\psi = (\boldsymbol{\rho}\boldsymbol{e}_x + \eta\tan\psi)\cos\psi.$$

Hence
$$\mu = \frac{1}{\sigma_1}\int_{\sigma_1} \exp\left[\mathrm{i}\langle k\rangle\left(\boldsymbol{\rho e}_x + \eta \tan\psi\right)\cos\psi\right]\omega\,\mathrm{d}\omega\,\mathrm{d}\theta,$$
where θ is the angle between the plane of the drawing and $\boldsymbol{\omega}$. As
$$\cos\psi = 1 - \frac{\psi^2}{2!} + \frac{\psi^4}{4!} - \dots$$
if
$$\langle k\rangle\left(\boldsymbol{\rho e}_x + \eta\tan\psi\right)\tfrac{1}{2}\psi^2 \leqslant 1,$$
$$\mu = \exp\left[\mathrm{i}\langle k\rangle\,\boldsymbol{\rho e}_x\right]\frac{1}{\sigma_1}\int_0^{\frac{1}{2}\mathscr{D}_1}\int_0^{2\pi}\left[\exp \mathrm{i}\langle k\rangle\,\eta\,\frac{\omega}{\langle n\rangle\,\mathscr{F}}\cos\theta\right]\omega\,\mathrm{d}\omega\,\mathrm{d}\theta,$$
because
$$\tan\psi = \frac{1}{\langle n\rangle}\frac{\omega}{\mathscr{F}}\cos\theta,$$
From here
$$\mu = \exp\left[\mathrm{i}\langle k\rangle\,\boldsymbol{\rho e}_x\right]\,2\,\frac{J_1(k_0\mathscr{D}_1\eta/2\mathscr{F})}{k_0\mathscr{D}_1\eta/2\mathscr{F}}.$$

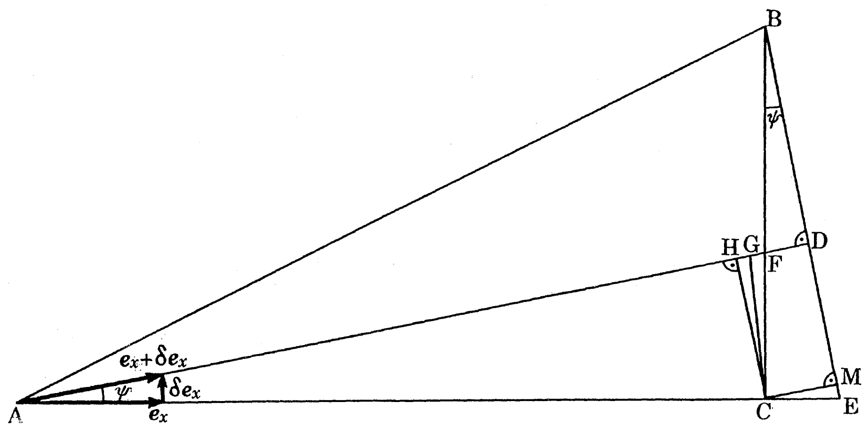

Figure 5

Appendix II

The convolution square of a circle
$$T(\boldsymbol{n}) = \begin{cases} 1 & (|\boldsymbol{n}| \leqslant \tfrac{1}{2}\mathscr{D}), \\ 0 & (|\boldsymbol{n}| > \tfrac{1}{2}\mathscr{D}). \end{cases}$$
The convolution square of the shape function
$$\overset{2}{\widetilde{T(\boldsymbol{n})}} = \iint_\sigma T(\boldsymbol{n})\,T(\boldsymbol{n}+\boldsymbol{\eta})\,\mathrm{d}\sigma$$
can be easily calculated by using the graphical interpretation of the convolution operation.†

Figure 6 shows the original function and its 'ghost' shifted by the vector $\boldsymbol{\eta}$. The shaded portion represents the area common to both of them and it is a measure of $\overset{2}{\widetilde{T(\boldsymbol{n})}}$ for a displacement $\boldsymbol{\eta}$.

An elementary calculation leads to the result
$$\overset{2}{\widetilde{T(\boldsymbol{n})}} = \tfrac{1}{2}\mathscr{D}^2 \operatorname{arc\,cos}(\eta/\mathscr{D}) - \tfrac{1}{2}\eta\sqrt{(\mathscr{D}^2-\eta^2)}$$

† The function $T(\boldsymbol{n})$ is symmetrical with respect to the origin. It can be shown (Hosemann & Bagchi 1962) that for such functions, called usually 'centrosymmetric', the convolution square is identical to the convolution product.

and, as
$$\mathscr{T}(\eta) = \frac{1}{\sigma}\overset{2}{\widetilde{T(n)}},$$

$$\mathscr{T}(\eta) = \frac{2}{\pi}\arccos\frac{\eta}{\mathscr{D}} - \frac{2}{\pi}\frac{\eta}{\mathscr{D}^2}\sqrt{(\mathscr{D}^2-\eta^2)}.$$

The convolution square of the one-dimensional shape function (Dirichlet's step function)

$$T(x) = \begin{cases} 1 & (|x-\tfrac{1}{2}\Delta l| \leqslant \tfrac{1}{2}\Delta l), \\ 0 & (|x-\tfrac{1}{2}\Delta l| > \tfrac{1}{2}\Delta l). \end{cases}$$

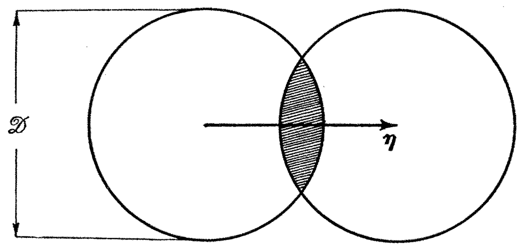

Figure 6

The convolution square of the one-dimensional shape function

$$\overset{2}{\widetilde{T(x)}} = \int_{-\infty}^{\infty} T(x)\, T(x+\xi)\,\mathrm{d}x$$

can be immediately calculated; it is easy to see that

$$\overset{2}{\widetilde{T(x)}} = \Delta l - \xi$$

and
$$\mathscr{T}(\xi) = \overset{2}{\widetilde{T(x)}}/\Delta l,$$

$$\mathscr{T}(\xi) = 1 - \xi/\Delta l,$$

i.e. the triangular function.

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